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# ON THE STABILITY OF EQUILIBRIUM POSITIONS IN NON-STATIONARY FORCE FIELDS*

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The stability of equilibrium positions is investigated for mechanical systems in force fields with potentials of the form $p(t)V$, where $V$ is a function of the generalized coordinates. Systems of this form are frequently encountered in applications. It is shown that if the factor $p(t)$ increases monotonically to $+\infty$ as $t\to+\infty$, then stability conditions for equilibria can be formulated in the form of extremal properties of the function $V$. The general results are applied to the problem of the motion of a rigid body in an infinite volume of an ideal fluid.

*1. Introduction.* Suppose that $x_1, \ldots, x_n$ are generalized coordinates for a mechanical system with $n$ degrees of freedom, $T$ is the kinetic energy and $-p(t)V(x)$ is the force function. The motion is described by Lagrange's equations

$$(\partial T/\partial x^{\cdot})^{\cdot} - \partial T/\partial x = -p\partial V/\partial x \tag{1.1}$$

We will always assume that $p(t) > 0$ for all values of $t$.

Equations of the form (1.1) are often encountered in applications. We shall give an example of intrinsic interest. Consider the problem of the motion of a heavy rigid body in a perfect fluid at rest at infinity and in irrotational motion /1/. We assume for simplicity that the rigid body has three mutually orthogonal planes of symmetry. In this case the kinetic energy of the "body plus fluid" system has the form

$$T = (A\omega,\ \omega)/2 + (Cv,\ v)/2$$

where $\omega$ is the angular velocity of the rigid body, $v$ the velocity of the point of intersection $O$ of the planes of symmetry, and $A$ and $C$ are symmetric positive definite matrices. Because of the assumed symmetry of the body, the resultant of the force of gravity and the buoyancy force acts on the point $O$. Let $P$ be the magnitude of the sum of these forces.

The motion of the rigid body can be represented in the form of a system of Kirchhoff equations /1/

$$(\partial T/\partial v)^{\cdot} + \omega\times(\partial T/\partial v) = -P\gamma,\quad (\partial T/\partial\omega)^{\cdot} + \omega\times(\partial T/\partial\omega) + v\times(\partial T/\partial v) = 0 \tag{1.2}$$

(where $\gamma$ is the vertical unit vector). Eq.(1.2) needs to be supplemented with the geometrical Poisson's equations

$$\alpha^{\cdot} + \omega\times\alpha = \beta^{\cdot} + \omega\times\beta = \gamma^{\cdot} + \omega\times\gamma = 0 \tag{1.3}$$

where $\alpha$ and $\beta$ are fixed unit vectors orthogonal to the vector $\gamma$.

The equations of motion have three integral constants of the motion:

$$(\partial T/\partial v,\ \alpha) = c_1\quad (\partial T/\partial v,\ \beta) = c_2\quad (\partial T/\partial v,\ \gamma) = c_3 - Pt \tag{1.4}$$

From (1.4) we obtain the equality $Cv = c_1\alpha + c_2\beta + (c_3 - Pt)\gamma$.

Using this relation, the second equation of (1.2) can be put in the form of Euler's equation

$$A\omega^{\cdot} + \omega\times A\omega = \alpha\times\partial W/\partial\alpha + \beta\times\partial W/\partial\beta + \gamma\times\partial W/\partial\gamma \tag{1.5}$$

$$W = (z,\ C^{-1}z)/2,\ z = c_1\alpha + c_2\beta + (c_3 - Pt)\gamma$$

Eqs.(1.3) and (1.5) describe the rotation of a rigid body about a fixed point in a non-stationary force field with force function $-W$. These equations are particularly simple for $c_1 = c_2 = c_3 = 0$:

$$A\omega^{\cdot} + \omega\times A\omega = P^2t^2(\gamma\times\partial V/\partial\gamma),\ \gamma^{\cdot} + \omega\times\gamma = 0 \tag{1.6}$$

$$V = (C^{-1}\gamma,\ \gamma)/2$$

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They are identical with the rotation equations for a rigid body in an axisymmetric force field with potential energy $P^2t^2V$. In the general case $W = P^2t^2V + tW_1 + W_2$, where $W_1$ and $W_2$ depend only on the position of the body.

The equilibrium positions of system (1.1) coincide with the critical points of the function $V$. It turns out that for a certain class of functions $p$ the stability of the equilibria is determined exclusively by the extremal properties of the function $V$.

Suppose $dV(0) = 0$ and $V(0) = 0$.

2. ***Stability conditions.*** *Theorem 1.* Suppose $x = 0$ is a local maximum of a smooth function $V$. Then the equilibrium $x = 0$ is unstable.

Theorem 1 generalizes the classical Lyapunov result for unstable equilibrium in a stationary field ($p = \text{const}$), when $V = V_m + \ldots$ and the homogeneous form $V_m$ reaches a strict maximum at the point $x = 0$. The proof is based on applying the results of /2/.

When the function $V$ is a homogeneous form of degree $m \geqslant 2$, Theorem 1 can be proved by Lyapunov-Chetayev methods.

As an illustration we consider the case of a constant matrix

$$\| \partial^2 T / \partial x_i^{\cdot} \partial x_j^{\cdot} \| = \| g_{ij} \| = G$$

For the moment of inertia $I = (Gx, x)$ we have the virial identity

$$I^{\cdot\cdot} = 4T - 2pmV$$

Because $V(x) \leqslant 0$ by assumption, $I^{\cdot\cdot} \geqslant 0$. If $I^{\cdot} = 2(Gx, x^{\cdot}) > 0$ at the initial instant, then $I(t) \to \infty$ as $t \to +\infty$. Hence the $x = 0$ equilibrium is unstable.

We make the change of variable

$$\tau = g(t), \quad g^{\cdot} = p^{1/2}$$

in Eqs.(1.1). It is clear that $g$ is a monotonic function of $t$. Denoting differentiation with respect to $\tau$ by a prime, we obtain

$$(\partial T^*/\partial x')' - \partial T^*/\partial x = -\partial V/\partial x - Gx'k(\tau) \tag{2.1}$$
$$T^* = \sum g_{ij}(x) x_i' x_j'/2, \quad k = p^{\cdot}/(2p^{3/2})$$

Applying the change of energy theorem to Eqs.(2.1), we obtain the equality

$$(T^* + V)' = -k(Gx', x') = -2kT^* \tag{2.2}$$

From (2.2) we immediately obtain the following proposition.

*Proposition 1.* Suppose $p^{\cdot} \geqslant 0$ and the function $V$ has a strict local minimum at the point $x = 0$. Then the $x = 0$ equilibrium is stable relative to the coordinates $x_1, \ldots, x_n$.

As will be shown below, there may be no stability relative to the velocities $x^{\cdot}$.

*Theorem 2.* Suppose the following conditions are satisfied:

$$1) \quad p^{\cdot} \geqslant 0, \quad 2) \lim_{t \to +\infty} p(t) = +\infty, \quad 3) \; p^{\cdot\cdot}p \leqslant {}^3/_2 p^{\cdot 2} \tag{2.3}$$

and the function $V$ is analytic in $x_1, \ldots, x_n$ in an neighbourhood of the point $x = 0$. Then if $V$ has a strict local minimum at the point $x = 0$, the $x = 0$ equilibrium is asymptotically stable relative to the coordinates $x_1, \ldots, x_n$ ($x_i \to 0$ as $t \to +\infty$). If, however, the function $V$ does not have a local minimum at the point $x = 0$, then the $x = 0$ equilibrium is unstable relative to the coordinates $x_1, \ldots, x_n$.

Condition 3 is equivalent to assuming that the function $k(\tau)$ diminishes monotonically as $\tau \to +\infty$. It is obviously satisfied by functions $p(t)$ of the form $ct^\alpha$, $c \exp(\alpha t)$ and $c \ln(\alpha t)$ (with $c, \alpha > 0$).

We remark that Theorem 2 does not hold in the case when the function $V$ is infinitely differentiable, but not analytic. Here is a simple example: $V(0) = 0$, $V(x) = \exp(-x^{-2}) \cos x^{-2}$ when $x \neq 0$. The equilibrium position $x = 0$ is not a local minimum of the function $V$. However, if conditions (2.3) are satisfied, this is a stable equilibrium with respect to the $x$ coordinate.

To prove Theorem 2 we introduce the function $H(x', x) = T^* + V$. Because $p^{\cdot} \geqslant 0$ by assumption, then in view of the identity (2.2) the function $H$ decreases monotonically. Consequently, as $\tau \to \infty$, either $H(\tau) \to -\infty$ or $H(\tau) \to c$.

*Theorem 3.* Suppose conditions (2.3) are satisfied and the trajectory of motion $x(t)$ is bounded. Then $c$ is a critical value of the function $H$.

Theorem 2 follows from Theorem 3 and the isolation properties of critical values of the analytic function $V$ /3/. We note that the critical points of the function $H$ are pairs ($x = x_0$, $x' = 0$) where $x_0$ is a critical point of the function $V$. Thus $c = V(x_0)$.

We assume that $H(\tau) \to c$ when $\tau \to +\infty$. From (2.2)

$$\int_{\tau_0}^{\infty} k(\tau)\, T^*(\tau)\, d\tau < \infty \tag{2.4}$$

We shall show that $c \leqslant \sup V$. Indeed, suppose $c > \sup V$. Because $T^* + V \geqslant c$, we have $T^* \geqslant c - V \geqslant c_1 > 0$. But then

$$\int_{\tau_0}^{\infty} kT^*\, d\tau \geqslant c_1 \int_{\tau_0}^{\infty} k\, d\tau = c_1 \int_{t_0}^{\infty} \frac{p^{\cdot}}{2p}\, dt = \frac{c_1}{2} \ln p \Big|_{t_0}^{\infty} = \infty$$

since $p(t) \to +\infty$ as $t \to +\infty$ (condition 2). This is a contradiction. We emphasise that here we have not used condition 3 on the monotonicity of the function $k(\tau)$.

In the general case, the same idea is used in the proof of Theorem 3. However, if $c < \sup V$, the function $T^*(\tau)$ can have positive values as small as desired and even vanish. We assume that $c$ is not a critical value of the function $H$, and hence of the function $V$. If for sufficiently large values of $\tau$ the changed kinetic energy $T^*(\tau)$ is small, then the value of $V(x(\tau))$ is very little different from $c$. However, near the hypersurface $\{V(x) = c\}$ there is no equilibrium position, and hence over the time $\Delta\tau \leqslant \varepsilon$ the kinetic energy $T^* \geqslant \varkappa > 0$. (A rigorous proof is performed using the method described in /4/). Because Theorem 3 assumes that the trajectory of motion $x(\tau)$ is bounded, $\varepsilon$ and $\varkappa$ are positive constants that are fixed for a given trajectory. Furthermore, the energy $T^*(\tau) \geqslant \varkappa$ over the time interval $\Delta\tau \geqslant \delta = \mathrm{const} > 0$.

One thus finds a sequence of new instants of time $\tau_0, \tau_1, \tau_2, \ldots$ such that $T^*(\tau) \leqslant \varkappa$ for $\tau \in [\tau_{2n}, \tau_{2n+1}]$ $(n = 0, 1, 2, \ldots)$ and $T^*(\tau) \geqslant \varkappa$ for $\tau \in [\tau_{2n-1}, \tau_{2n}]$ $(n = 1, 2, \ldots)$.

As has already been noted, $\tau_{2n+1} - \tau_{2n} \leqslant \varepsilon$ and $\tau_{2n} - \tau_{2n-1} \geqslant \delta$ for all values of $n$. If the sequence $\tau_n$ is bounded, then $T^*(\tau) \geqslant \varkappa$ for all $\tau \geqslant \max \tau_n$ and hence integral (2.4) diverges.

The most interesting case is that of an infinite sequence $\tau_n$.

*Lemma.* We assume that the positive function $f(\tau)$ decreases monotonically and that

$$\int_{\tau_0}^{\infty} f(\tau)\, d\tau = \infty$$

Then for every monotonic sequence $\tau_n$ such that $\tau_{2n+1} - \tau_{2n} \leqslant \varepsilon$ and $\tau_{2n} - \tau_{2n-1} \geqslant \delta$, we have the equality

$$\sum \int_{\tau_{2n-1}}^{\tau_{2n}} f(\tau)\, d\tau = \infty \tag{2.5}$$

(where summation runs over $n$ from 1 to $\infty$).

*Proof.* Inserting, if necessary, new intervals of zero length into the sequence of intervals $[\tau_{2n}, \tau_{2n+1}]$, one can arrange that $\tau_k \leqslant k\mu + \sigma$ for some positive $\mu$ and $\sigma$. In view of the monotonicity of the function $f$ we have the inequality

$$\int_{\tau_{2n-1}}^{\tau_{2n}} f(\tau)\, d\tau \geqslant f(\tau_{2n})(\tau_{2n} - \tau_{2n-1}) \geqslant \delta f(\tau_{2n})$$

Consequently, the sum (2.5) is not less than $\delta \sum f(\tau_{2n})$. Because $\tau_{2n} \leqslant 2\mu n + \sigma$ and $f$ decreases monotonically, the sum series (2.5) is not less than $\delta \sum f(2\mu n + \sigma)$. However, this series diverges to $+\infty$ by the Cauchy-Maclaurin integral test.

In the intervals $[\tau_{2n-1}, \tau_{2n}]$ we have the inequality $T^* \geqslant \varkappa$. Consequently,

$$\int_{\tau_0}^{\infty} kT^* d\tau \geqslant \varkappa \sum \int_{\tau_{2n-1}}^{\tau_{2n}} k(\tau)\, d\tau$$

However, in view of the lemma the latter sum is equal to $\infty$. This contradicts the assumption of the convergence of integral (2.4). Theorem 3 (together with Theorem 2) is proved.

**3. *Application to the problem of the motion of a rigid body in an ideal fluid.*** Eqs.(1.6) have an area integral $(A\omega, \gamma) = c$. By fixing the value of $c$ one can reduce the number of degrees of freedom from three to two. If $c = 0$, then the reduced system is natural. The configuration space is the Poisson sphere $S^2 = \{(\gamma, \gamma) = 1\}$, and the potential energy is identical with the restriction of the function $P^2 t^2 (C^{-1}\gamma, \gamma)/2$ to the sphere $S^2$. This function has six critical points - the equilibrium positions of the reduced system. They correspond to the unit eigenvectors of the symmetric operator $C$. We denote them by $\pm e_1, \pm e_2, \pm e_3$. Suppose $(C^{-1}e_1, e_1) \geqslant (C^{-1}e_2, e_2) \geqslant (C^{-1}e_3, e_3)$, so that the function $V$ has its minimum values at the points $\gamma = \pm e_3$. If this is a strict minimum, then the equilibrium positions are asymptotically stable with respect to the coordinates on $S^2$, while the remaining four equilibria are unstable. One can prove that for almost all motions $\gamma(t) \to \pm e_3$ for $t \to \infty$.

In order to give a mechanical interpretation of this result, we consider the translational motion of a rigid body with unit velocity $v$. The kinetic energy of the motion is $(Cv, v)/2$ and is a maximum if $v = \pm e_3$. Hence, if the body moves in the direction of the vector $\pm e_3$, then its virtual mass has its maximum value. In particular, if the rigid body has the form of a lamina, then the vector $\pm e_3$ is orthogonal to its plane. Thus under the given assumptions a freely falling heavy rigid body in a perfect fluid asymptotically tends to a position in which the axis with the greatest virtual mass is vertical. If the rigid body is plane, then this plane tends towards a horizontal position.

In the general case, when the integral constants of the motion $c_1$ and $c_2$ are non-zero, the motion is described by Eqs.(1.3) and (1.5). In local coordinates on the $SO$ (3) group they can be put in the form of Eqs.(2.1):

$$(\partial T^*/\partial x')' - \partial T^*/\partial x = -\partial W^*/\partial x - Gx'/(2\tau)$$
$$(\tau = t^2/2,\ W^* = P^2 V + W_1/(2\tau)^{1/2} + W_2/(2\tau))$$

We introduce the function $H = T^* + W^*$ and study its behaviour as $\tau \to \infty$. Because $H' = -\partial H/\partial\tau - T^*/\tau$,

$$H' = W_1/(2\tau)^{3/2} + W_2/(2\tau^2) - T^*/\tau \tag{3.1}$$

We assume that the motion trajectory $x(t)$ is bounded. This condition is certainly satisfied in the motion of the rigid body because of the compactness of the $SO$ (3) group. Hence the functions $W_1(x(\tau))$ and $W_2(x(\tau))$ are bounded. Hence the improper integrals

$$\int_{\tau_0}^{\infty} \frac{W_1}{(2\tau)^{3/2}}\, d\tau,\ \int_{\tau_0}^{\infty} \frac{W_2}{2\tau^2}\, d\tau$$

converge. Using the boundedness of the function $H(\tau)$ from (3.1), we obtain the convergence of the integral

$$\int_{\tau_0}^{\infty} \frac{T^*}{\tau}\, d\tau \tag{3.2}$$

But then from (3.1) we find that

$$\lim_{\tau\to\infty} H(\tau) = c$$

exists.

Applying the discussion from Sect.2 and using the result for the convergence of integral (3.2), one can show that the constant $c$ is identical with the critical value of the function $V$. In particular, if the point $x = 0$ is a strict local minimum of an analytic function $V$, the motion $x(t)$ with small initial data $x(0)$ and $x^{\cdot}(0)$ tends to the point $x = 0$ as $t \to \infty$.

In the problem under consideration the function $V$ has no isolated minima: it takes minimum values at two circles given by the equations $\gamma = \pm e_3$. Hence as $t \to \infty$ a falling body tends in general to a position such that the axis with the largest virtual mass is vertical. If $c_1$ and $c_2$ are non-zero, the body can rotate about this axis.

We again return to the general problem of the motion of a mechanical system with potential energy $W = t^2 V + tW_1 + W_2$. If the critical points of the functions $V, W_1$ and $W_2$ do not coincide, then the equations of motion have no positions of equilibrium. However, given certain general assumptions, there are instead some remarkable special solutions to the equations of motion.

*Proposition 2.* If $x = 0$ is a non-degenerate critical point of the function $V$, the equations of motion have a solution $x(t)$ that can be asymptotically represented by the expansion

$$\sum_{m=1}^{\infty} \frac{x^{(m)}}{t^m}, \quad x^{(m)} \in R^n \tag{3.3}$$

This means that as $t \to \infty$,

$$x(t) - \sum_{m=1}^{N} \frac{x^{(m)}}{t^m} = O\left(\frac{1}{t^{N+1}}\right)$$

In particular, $x(t) \to 0$ as $t \to \infty$.

*Proof*. The equations of motion have a unique formal solution in the form of the power series (3.3). The coefficient of $x^{(1)}$ is given by the equation

$$Cx^{(1)} + a = 0, \; C = \partial^2 V/\partial x^2 (0), \; a = \partial W_1/\partial x (0)$$

and the remaining coefficients are found by induction. The radius of convergence of series (3.3) is, as a rule, equal to zero. However, according to /5/, the equations of motion must have a solution for which the series (3.3) is the asymptotic expansion as $t \to \infty$.

In the general problem of a falling rigid body all the critical points of the function $V$ are degenerate. However, Proposition 2 can be applied to the special case of the motion of a rigid body when a plane of symmetry is in the vertical position.

*4. **Asymptotic forms of small oscillations.*** Consider the problem of the motion of system (1.1) in the neighbourhood of a stable equilibrium position $x = 0$. We shall suppose that the conditions of Proposition 1 are satisfied. The linearized equations of motion take the form

$$Ax^{\cdot\cdot} + p(t)\, Cx = 0 \tag{4.1}$$
$$(A = \| g_{ij}(0) \|, \; C = \partial^2 V/\partial x^2 (0))$$

With a linear transformation of the coordinates $x$ the matrix $A$ can be reduced to unity and $C$ diagonalized: $\operatorname{diag}[\omega_1^2, \ldots, \omega_n^2]$.

We shall consider a typical case, when all the $\omega_k > 0$. In the new coordinates $x_1, \ldots, x_n$ Eq.(3.1) separates:

$$x_k^{\cdot\cdot} + p\omega_k^2 x_k = 0, \; k = 1, \ldots, n \tag{4.2}$$

We shall study the behaviour of solutions of this linear equation as $t \to \infty$ (dropping the index $k$).

Replacing the time coordinate $t \to \tau$ using the formula $\tau^{\cdot} = p^{1/2}$ and making the substitution $x = zp^{-1/4}$, we reduce Eq.(4.2) to the form

$$z'' + q(\tau) z = 0, \; q = \omega^2 - k'/2 - k^2/4 \tag{4.3}$$

If the conditions

$$1) \quad q > 0 \quad \text{when} \quad \tau > \tau_0, \; 2) \lim_{\tau \to \infty} q(\tau) > 0, \; 3) \int_{\tau_0}^{\infty} | q'(\tau) | \, d\tau < \infty \tag{4.4}$$

are satisfied, then the solution of (4.3) can be represented in the form

$$z = c_1 \sin \int_{\tau_0}^{\tau} (q(u))^{1/2} du + c_2 \cos \int_{\tau_0}^{\tau} (q(u))^{1/2} du + \xi(\tau)$$

where $c_1$ and $c_2$ are arbitrary constants, and $\xi$ and $\xi' \to 0$ as $\tau \to \infty$ (/6, Chapter V/). For example, for $p = \alpha^2 t^{2\beta}$ $(\alpha, \beta = \text{const})$, the value of $q(\tau)$ is $\omega^2 + c/\tau^2$ $(c = \text{const})$ and so all the conditions of (4.4) are clearly satisfied.

Since $x = zp^{-1/4}$, the $x$ coordinate performs oscillations whose amplitude decreases as $(p(t))^{-1/4}$, while the frequency increases with time. For example, for a power-law time dependence of $p$ the frequency can be represented in the form of the sum

$$\omega \int_{t_0}^{t} (p(u))^{1/2} du + \eta(t)$$

where the function $\eta(t)$ is bounded.

The coordinates $x_1, \ldots, x_n$ can be called normal, and the motions described by the linear Eqs.(4.2) can be called normal oscillations. The general solution of system (4.1) is the sum of normal oscillations whose amplitude decreases without limit in time. For example, in the problem of a falling rigid body in a fluid the amplitude decreases as $t^{-1/2}$, while the frequency increases as $t$.

*5. The influence of gyroscopic forces.* We add the term $\Gamma x^{\cdot}$ to the right-hand side of Eq.(1.1), where $\Gamma$ is a skew-symmetric matrix. The presence of gyroscopic forces does not, of course, change the equilibrium positions and the identity (2.2) remains valid. Hence Proposition 1 is still true in the more-general case. However, the asymptotic forms of small oscillations will be very different.

We will consider as an example the linear system of equations

$$x^{\cdot\cdot} + \omega y^{\cdot} = -p(t)\,x, \quad y^{\cdot\cdot} - \omega x^{\cdot} = -p(t)\,y, \quad \omega = \text{const} \tag{5.1}$$

These equations admit an integral $x^{\cdot}y - xy^{\cdot} + \omega(x^2 + y^2)/2 = c$ which takes the form $r^2\varphi^{\cdot} = -\omega r^2/2 + c$ in polar coordinates $r, \varphi$.

We will investigate a particular solution corresponding to the case $c = 0$. Because $\varphi^{\cdot} = -\omega/2 = \text{const}$, the point moves along a line rotating with constant angular velocity around the origin of coordinates. From the change of kinetic energy theorem we obtain the equality

$$(r^{\cdot 2} + r^2\varphi^{\cdot 2})^{\cdot} + p\,(r^2)^{\cdot} = 0 \tag{5.2}$$

Using the relation $\varphi^{\cdot} = -\omega/2$, we obtain from (5.2) the equation

$$r^{\cdot\cdot} + (p + \omega^2/4)\,r = 0$$

If $p$ increases monotonically to infinity, then, according to Sect.4, the variable $r$ performs oscillations whose amplitude decreases without limit as $(p(t) + \omega^2/4)^{-1/4}$. In particular, $x^2 + y^2 \to 0$ as $t \to \infty$.

This last property is not accidental. It turns out that if the conditions (2.3) are satisfied and the point $x = 0$ is a strict local minimum of an analytic function $V$, then in presence of gyroscopic forces the $x = 0$ equilibrium is asymptotically stable with respect to the coordinates $x_1, \ldots, x_n$.

Gyroscopic forces are known to appear when the order of the system is reduced by a symmetry group. As an example, we again turn to Eqs.(1.6), which have a Noether integral $(A\omega, \gamma) = c$. For a fixed value of $c$, the rotation of the rigid body is described by a dynamical system with two degrees of freedom, under the influence of gyroscopic forces and additional potential forces with potential $c^2/[2(A\gamma, \gamma)]$. The presence of the additional potential does not change the equilibrium positions of the reduced system in this case. These particular motions can be treated as relative equilibria: one of the eigendirections of the operator $C$ (determined by vectors $e_1, e_2, e_3$ in Sect.3) is vertical and the body rotates about this axis with constant velocity. There are six of them in all, so in the general case (when the eigenvalues of the operator $C$ are different), two of them are asymptotically stable with respect to the $\gamma$ coordinate, while four of them are unstable.

In the general case, after the order has been reduced the reduced potential of the form $t^2V(x) + W(x)$ may not have critical points that are relative equilibria. However, in this case one can establish the existence of particular solutions of the form (3.3) to replace them, given the condition that the critical points of the function $V$ are non-degenerate.

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